MAT 303 - Calculus IV with Applications

Practice Final Exam

Stony Brook University
Fall 2023

[20 pts] **Problem 1.**

[5 pts] (i) Find the solution of the following initial value problem and give the interval of existence.

$$y'(x) + 1 = 2y(x), \quad y(0) = 1.$$

[5 pts] (ii) Find the solution of the following initial value problem and give the interval of existence.

$$y'(x) = 1 + x + y(x) + xy(x), \quad y(0) = 0.$$

[5 pts] (iii) Find the general solution and any singular solution of the ODE

$$y' = 3\sqrt{xy}.$$

[5 pts] (iv) Explain why there exists a unique local solution of the following initial value problem.

$$y' = x \ln y, \quad y(1) = 1.$$

[20 pts] **Problem 2.**

[5 pts] (i) Examine whether the following ODE is exact.

$$(2x + 3y)dx + (3x + 2y)dy = 0$$

[5 pts] (ii) Solve the above equation.

[5 pts] (iii) Give an initial condition for $y(0)$ so that the above initial value problem has a unique local solution.

[5 pts] (iv) With the value of $y(0)$ that you prescribed in part (iii) use Euler's method with 2 steps to approximate $y(1)$.

[15 pts] **Problem 3.** Find the general solutions of the following ODEs.

[5 pts] (i) $6y^{(4)} + 11y'' + 4y = 0.$

[5 pts] (ii) $y'' - y' - 2y = 3x + 4.$

[5 pts] (iii) $y^{(3)} + y' = 2 - \sin x.$

[20 pts] **Problem 4.** Consider the system $x' = -y$, $y' = 1.01x - 0.2y$.

[5 pts] (i) Write the system in matrix form $\mathbf{x}' = A\mathbf{x}$ and find its general solution using the eigenvalue method. (Note that $x(t), y(t)$ denote (real valued) functions but $\mathbf{x}(t)$ denotes a vector valued function.)

[5 pts] (ii) Use the above solution to compute the matrix exponential e^{At}.

[5 pts] (iii) Solve the initial value problem $x(0) = 0$ and $y(0) = 1$. That is, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

[5 pts] (iv) Draw the trajectories of the solution of part (iii). Use arrows to indicate the behavior of the solution as $t \to \infty$.

[15 pts] **Problem 5.** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

[5 pts] (i) Find the eigenvalues of A, their multiplicity, and defect.

[5 pts] (ii) Find the general solution of $\mathbf{x}' = A\mathbf{x}$ by finding generalized eigenvectors of A.

[5 pts] (iii) Find the general solution of $\mathbf{x}' = A\mathbf{x}$ by computing the matrix exponential e^{At}.

[15 pts] **Problem 6.** Study textbook problems on the following topics:
Natural growth models, population models, acceleration-velocity models, (forced) oscillations and resonance, endpoint problems.